REMARKS

Applicant respectfully requests reconsideration and further examination of this application.

Applicant has amended the equation on page 8 of the specification to be "sinc" instead of "sine". This corrects what clearly was only a typographical error, as it is well know in the field that sinc(x) = sin(x) / (x) (see attached Exhibit A).

Applicant has amended Claims 15, 16, and 18-26 to be dependent upon independent Claim 14 and Claim 17 to be dependent upon intervening Claim 16, to correct what were clearly only typographical errors.

Applicant believes that the application is in condition for allowance and requests the same.

Dated: 6/23/04

Respectfully submitted,

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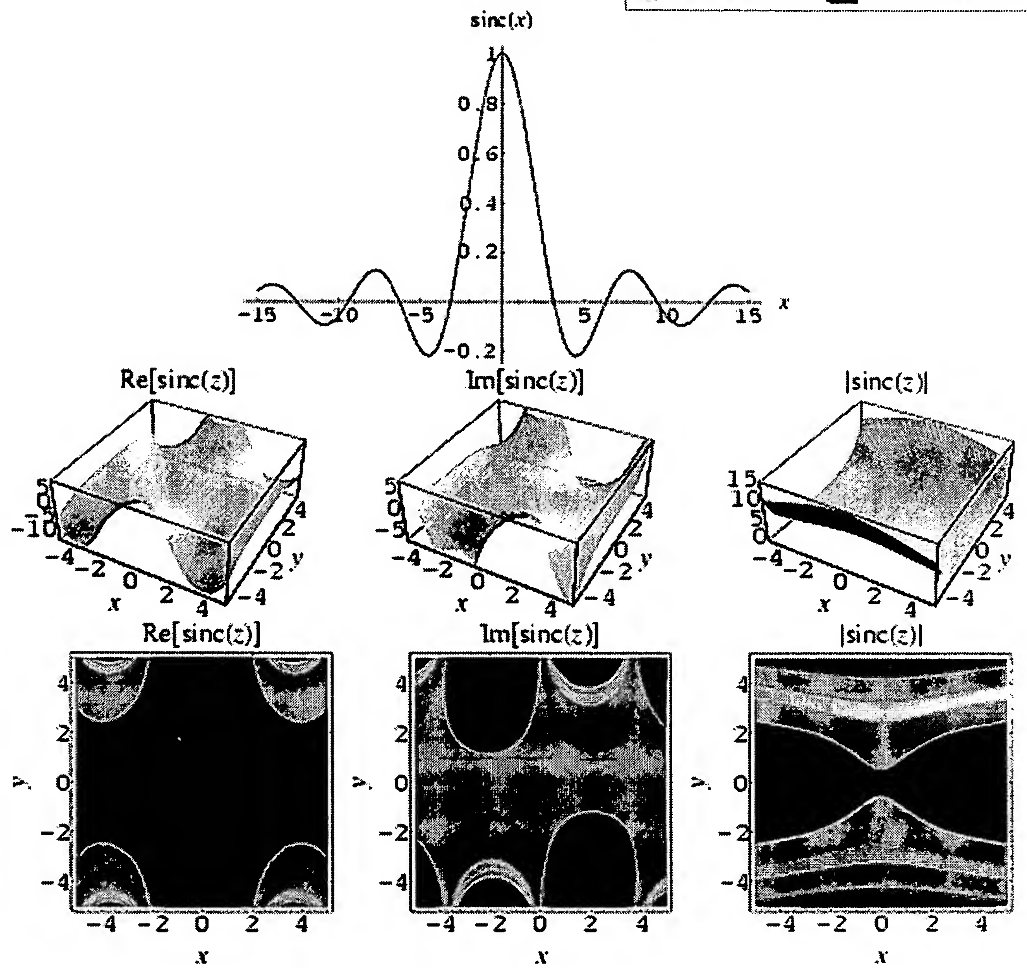
Sinc Function



The sinc function $\text{sinc}(x)$, also called the "sampling function," is a function that arises frequently in signal processing and the theory of Fourier transforms. The full name of the function is "sine cardinal," but it is commonly referred to by its abbreviation, "sinc." There are two definitions in common use. The one adopted in this work defines

$$\text{sinc}(x) \equiv \begin{cases} 1 & \text{for } x = 0 \\ \frac{\sin x}{x} & \text{otherwise,} \end{cases} \tag{1}$$

where $\sin x$ is the sine function. This has the normalization

$$\int_{-\infty}^{\infty} \text{sinc}(x)\, dx = \pi, \tag{2}$$

EXHIBIT
A
(6 pages)

while Woodward (1953), McNamee *et al.* (1971), and Bracewell (1999, p. 62) adopt the

alternative definition

$$\operatorname{sinc}_\pi(x) \equiv \begin{cases} 1 & \text{for } x = 0 \\ \frac{\sin(\pi x)}{(\pi x)} & \text{otherwise.} \end{cases} \tag{3}$$

The latter definition is sometimes more convenient as a result of its simple normalization,

$$\int_{-\infty}^{\infty} \operatorname{sinc}_\pi(x)\, dx = 1. \tag{4}$$

The sinc function is closely related to the spherical Bessel function of the first kind $j_n(x)$ and, in particular,

$$\operatorname{sinc}(x) = j_0(x). \tag{5}$$

Let $\Pi(x)$ be the rectangle function, then the Fourier transform of $\Pi(x)$ is the sinc function

$$\mathcal{F}_x[\Pi(x)](k) = \operatorname{sinc}(\pi k). \tag{6}$$

The sinc function therefore frequently arises in physical applications such as Fourier transform spectroscopy as the so-called instrument function, which gives the instrumental response to a delta function input. Removing the instrument functions from the final spectrum requires use of some sort of deconvolution algorithm.

The sinc function can be written as a complex integral by noting that, for $x \neq 0$,

$$\begin{aligned} \operatorname{sinc}(nx) &\equiv \frac{\sin(nx)}{nx} = \frac{1}{nx}\frac{e^{inx} - e^{-inx}}{2i} \\ &= \frac{1}{2inx}[e^{itx}]_{-n}^{n} = \frac{1}{2n}\int_{-n}^{n} e^{ixt}\, dt, \end{aligned} \tag{7}$$

and that $\operatorname{sinc}(nx)$ and the integral both equal 1 for $x = 0$.

The sinc function can also be written as the infinite product

$$\operatorname{sinc} x = \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right), \tag{8}$$

a result discovered in 1593 by Francois Viète (Kac 1959, Morrison 1995).

Definite integrals involving the sinc function include

$$\int_0^\infty \operatorname{sinc}(x)\, dx = \tfrac{1}{2}\pi \tag{9}$$

$$\int_0^\infty \operatorname{sinc}^2(x)\, dx = \tfrac{1}{2}\pi \tag{10}$$

$$\int_0^\infty \operatorname{sinc}^3(x)\, dx = \tfrac{3}{8}\pi \tag{11}$$

$$\int_0^\infty \operatorname{sinc}^4(x)\,dx = \tfrac{1}{3}\pi \quad (12)$$

$$\int_0^\infty \operatorname{sinc}^5(x)\,dx = \tfrac{115}{384}\pi. \quad (13)$$

After dividing out the constant factor of π, the values are 1/2, 1/2, 3/8, 1/3, 115/384, 11/40, 5887/23040, 151/630, 259723/1146880, ... (Sloane's A049330 and A049331; Grimsey 1945, Medhurst and Roberts 1965). These are all special cases of the amazing general result

$$\int_0^\infty \frac{\sin^a x}{x^b}\,dx = \frac{\pi^{1-c}(-1)^{\lfloor (a-b)/2\rfloor}}{2^{a-c}(b-1)!}\sum_{k=0}^{\lfloor a/2\rfloor - c}(-1)^k\binom{a}{k}(a-2k)^{b-1}[\ln(a-2k)]^c, \quad (14)$$

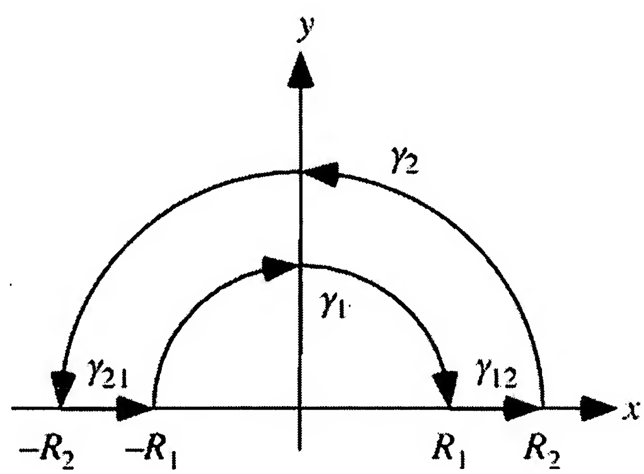
where *a* and *b* are positive integers such that $a \geq b > c$, $c \equiv a-b \pmod 2$, $\lfloor x\rfloor$ is the floor function, and 0^0 is taken to be equal to 1 (Kogan). This spectacular formula simplifies in the special case when *n* is a positive even integer to

$$\int_0^\infty \frac{\sin^{2n} x}{x^{2n}}\,dx = \frac{\pi}{2(2n-1)!}\left\langle {2n-1 \atop n-1} \right\rangle, \quad (15)$$

where $\left\langle {n \atop k} \right\rangle$ is an Eulerian number (Kogan). The solution of the integral can also be written in terms of the recurrence relation for the coefficients

$$c(a,b) = \begin{cases} \frac{\pi}{2^{a+1-b}}\binom{a-1}{\frac{1}{2}(a-1)} & \text{for } b=1 \text{ or } b=2 \\ \frac{a[(a-1)c(a-2,b-2)-a\cdot c(a,b-2)]}{(b-1)(b-2)} & \text{otherwise} \end{cases} \quad (16)$$

(Zimmerman 1977).



The half-infinite integral of $\operatorname{sinc}(x)$ can be derived using contour integration. In the above figure, consider the path $\gamma \equiv \gamma_1 + \gamma_{12} + \gamma_2 + \gamma_{21}$. Now write $z = Re^{i\theta}$. On an arc, $dz = iRe^{i\theta}\,d\theta$ and on the *x*-axis, $dz = e^{i\theta}\,dR$. Write

$$\int_{-\infty}^\infty \operatorname{sinc} x\,dx = \Im\int_\gamma \frac{e^{iz}}{z}\,dz, \quad (17)$$

where $\Im$ denotes the imaginary part. Now define

$$\begin{aligned} I &\equiv \int_\gamma \frac{e^{iz}}{z}\,dz \\ &= \lim_{R_1\to 0}\int_\pi^0 \frac{\exp(iR_1 e^{i\theta})}{R_1 e^{i\theta}}\, i\theta R_1 e^{i\theta}\,d\theta + \lim_{R_1\to 0}\lim_{R_2\to\infty}\int_{R_1}^{R_2}\frac{e^{iR}}{R}\,dR \\ &\quad + \lim_{R_2\to\infty}\int_0^\pi \frac{\exp(iz)}{z}\,dz + \lim_{\substack{R_1\to 0\\ R_2\to\infty}}\int_{R_2}^{R_1}\frac{e^{-iR}}{-R}\,(-dR), \end{aligned} \tag{18}$$

where the second and fourth terms use the identities $e^{i0}=1$ and $e^{i\pi}=-1$. Simplifying,

$$\begin{aligned} I &= \lim_{R_1\to 0}\int_\pi^0 \exp(iR_1 e^{i\theta})\,i\theta\,d\theta + \int_{0^+}^\infty \frac{e^{iR}}{R}\,dR \\ &\quad + \lim_{R_2\to\infty}\int_0^\pi \frac{\exp(iz)}{z}\,dz + \int_\infty^{0^+}\frac{e^{-iR}}{-R}\,(-dR) \\ &= -\int_0^\pi i\theta\,d\theta + \int_{0^+}^\infty \frac{e^{iR}}{R}\,dR + 0 + \int_{-\infty}^{0^-}\frac{e^{iR}}{R}\,dR, \end{aligned} \tag{19}$$

where the third term vanishes by Jordan's lemma. Performing the integration of the first term and combining the others yield

$$I = -i\pi + \int_{-\infty}^\infty \frac{e^{iz}}{z}\,dz = 0. \tag{20}$$

Rearranging gives

$$\int_{-\infty}^\infty \frac{e^{iz}}{z}\,dz = i\pi, \tag{21}$$

so

$$\int_{-\infty}^\infty \frac{\sin z}{z}\,dz = \pi. \tag{22}$$

The same result is arrived at using the method of complex residues by noting

$$\begin{aligned} I &= 0 + \tfrac{1}{2}2\pi i \operatorname*{Res}_{z=0} f(z) \\ &= i\pi\,(z-0)\frac{e^{iz}}{z}\bigg|_{z=0} = i\pi[e^{iz}]_{z=0} \\ &= i\pi, \end{aligned} \tag{23}$$

so

(24)

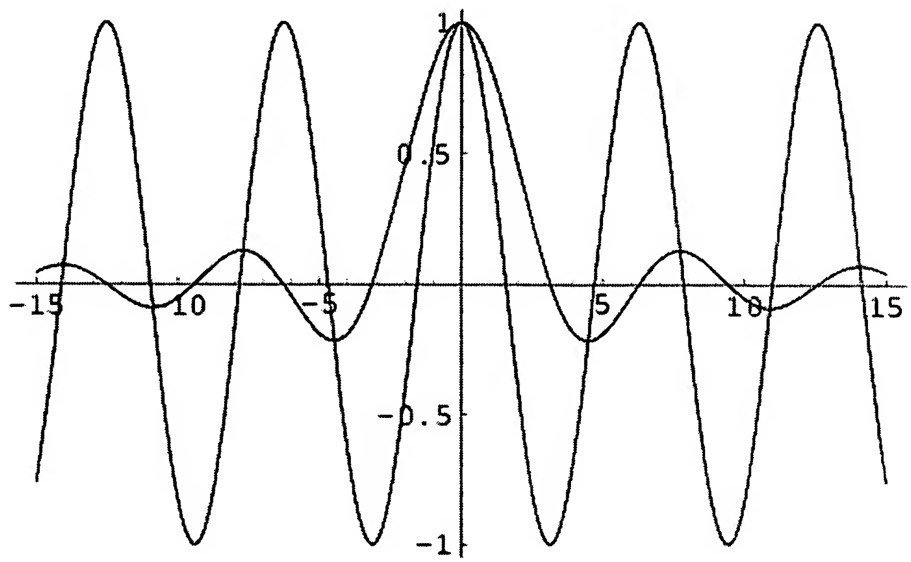
$$\Im(I) = \pi.$$

Since the integrand is symmetric, we therefore have

$$\int_0^\infty \frac{\sin x}{x}\,dx = \tfrac{1}{2}\pi, \qquad (25)$$

giving the sine integral evaluated at 0 as

$$\mathrm{si}(0) = -\int_0^\infty \frac{\sin x}{x}\,dx = -\tfrac{1}{2}\pi. \qquad (26)$$



An interesting property of $\mathrm{sinc}(x)$ is that the set of local extrema of $\mathrm{sinc}(x)$ corresponds to its intersections with the cosine function $\cos(x)$, as illustrated above.

SEE ALSO: Fourier Transform, Fourier Transform--Rectangle Function, Instrument Function, Jinc Function, Kilroy Curve, Sine, Sine Integral, Sinhc Function, Tanc Function

PAGES LINKING HERE: search

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